WEAK* SUPPORT POINTS OF CONVEX SETS IN E^*

BY

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ABSTRACT

Density theorems are obtained for weak* support points and support functionals of certain convex subsets of the dual of a Banach space.

Suppose that C is a closed convex nonempty subset of a real topological vector space E. A point x of C is said to be a *support point* of C if there exists a closed hyperplane H containing x such that C is on one side of H. Equivalently, x is a support point of C provided there exists a nontrivial continuous linear functional f on E such that $f(x) = \sup\{f(y): y \in C\} \equiv \sup f(C)$. (Such a functional is called a *support functional* of C.) Unless there are further hypotheses on the set C or the space E, there might be no support points in C. For instance, Klee [3] has shown the existence of a nonempty closed convex unbounded subset C of a certain metrizable, separable complete locally convex space (the space of all real sequences, in the product topology) such that C contains no support points. On the other hand, if E is a Banach space, then it is known [1] that the support points of C are dense in the boundary of C and that the support functionals of C are norm dense among those which are bounded above on C. (Furthermore, the latter assertion is valid in a normed linear space E *only* if E is complete.) It remains an open question whether a *bounded* closed convex subset of a metrizable complete locally convex space necessarily has support points. In the present note, we will prove two results, analogous to the two main theorems of [1], for certain special locally convex spaces, namely, the space E^* in its weak* topology, where E is a Banach space. (In its *norm topology*, of course, E^* is a Banach space and the results from [1] are applicable; they also apply in the weak* topology if E is reflexive.)

Thus, the dual of E^* is E itself, so that our closed hyperplanes will be weak* closed and they will be determined by functionals of the form $f \to f(x)$, for some $x \neq 0$ in E. It follows, then, that a support point (which, for emphasis, we will call a *weak** support point) of a weak* closed convex subset C of E^* is an element f of C such that $f(x) = \sup\{g(x): g \in C\} \equiv \sup C(x)$ for some $x \neq 0$ in E. Such an element x will be called a *weak* support functional* of C. The first of our two theorems may then be stated as follows.

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THEOREM 1. *If E is a Banach space and C is a weak* closed convex subset of E^* then the weak* support points of C are norm-dense in the norm-boundary of C.*

[It is easily seen that the choice of topologies (*norm*-dense in the *norm*-boundary) in the conclusion to this result is the best among the four possible combinations of norm and weak* topologies.]

Our second theorem is not as simple, but it has some simply stated corollaries. Geometrically, it says that if C can be strictly separated from a bounded weak* closed set A by a weak* closed hyperplane H, then it can be strictly separated from A by one of its weak* supporting hyperplanes H_1, with H_1 nearly parallel to H. There is no loss in generality in assuming that A is convex.

THEOREM 2. *Suppose that E is a Banach space, that C and A are weak* closed convex subsets of E^*, that A is bounded and that for some x in E $\sup C(x) < \inf A(x)$. Then for any $\varepsilon > 0$ there exists a weak* support functional, x_0 of C such that $\|x - x_0\| < \varepsilon$ and $\sup C(x_0) < \inf A(x_0)$.*

COROLLARY 1. *If E is a Banach space and C is a weak* closed convex subset of E^*, then the weak* support functionals of C are dense in E among those x for which* $\sup C(x) < \infty$.

A weak* *supporting half space* for C is a set of the form $\{g: g(x) \leqq \sup C(x)\}$ where x is a weak* support functional for C.

COROLLARY 2. *If E is a Banach space and C is a weak* closed convex subset of E^*, then C is the intersection of its weak* supporting half-spaces.*

The proofs of these corollaries are immediate from Theorem 2.

At the end of this note we will discuss examples which show that Theorem 1 may fail (even for bounded sets) if C is merely assumed to be norm closed, or if E is not assumed to be complete. Also, the fact that weak* closed convex subsets of E^* have a certain local weak* compactness property clearly does *not* imply that every weak* continuous functional which is bounded above on C is a support functional; consider the convex set defined by a hyperbola in the plane.

The methods of proof are closely related to those in [1], with one exception. In [1], the final step in producing support points was an application of the separation theorem in E; in the present note, it is applied in $E \times R$ to convex sets associated with the graphs of appropriate convex functions. This idea was suggested by A. Brøndsted and R. T. Rockafellar [2], who used it in their work on support points of the graphs of convex functions.

The first step in our proof is a lemma which is a bit more general than the corresponding lemma in [1] and less general than the one in [4]. We include the proof for the sake of completeness.

LEMMA 1. *Suppose that ϕ is an upper semicontinuous function on the Banach space E, with $-\infty \leqq \phi < \infty$. If $k>0$, partially order the set $X=\{x:\phi(x)>-\infty\}$ by*
(1) $x \succ y$ if and only if $k\|x-y\| \leqq \phi(x)-\phi(y)$. *If ϕ is bounded above on X then for any z in X there exists a maximal element x_0 in X such that $x_0 \succ z$.*

Proof. Since the ordering defined in (1) is *proper*, we can apply Zorn's lemma to the set $Z=\{x:x\in X,\ x\succ z\}$. If W is any linearly ordered subset of Z, let $a=\sup\{\phi(w): w\in W\}$; this is finite since ϕ is bounded above on X. Since W is linearly ordered we can choose a sequence $\{w_n\}\subset W$ with $w_{n+1}\succ w_n$ for $n=1,2,3,\cdots$ such that $\phi(w_n)\to a$. It follows that if $m\geqq n$, then $w_m \succ w_n$ and hence $k\|w_m-w_n\|\leqq \phi(w_m)-\phi(w_n)$. Since $\phi(w_n)$ converges, this shows that $\{w_n\}$ is a Cauchy sequence in E and hence converges to an element x in E. Since ϕ is upper semicontinuous, $\phi(x)\geqq \limsup\phi(w_n)=a$, so $x\in X$. To see that x is an upper bound for W, first note that for any n,

$$k\|x-w_n\| = \lim_{m\geqq n} k\|w_m-w_n\| \leqq \limsup_{m\geqq n}\phi(w_m)-\phi(w_n) = a-\phi(w_n) \leqq$$
$$\leqq \phi(x)-\phi(w_n),$$

so $x\succ w_n$. Finally, for any w in W, $w\neq x$, there exists n with $w_n\succ w$. (If not, then for all n, $k\|w-w_n\|\leqq \phi(w)-\phi(w_n)\leqq a-\phi(w_n)\to 0$, hence $w=\lim w_n = x$.) Thus, Zorn's lemma is applicable and Z has a maximal element x_0. If $x\in X$ and $x\succ x_0$, then $x\succ x_0\succ z$ so $x\in Z$ and hence $x=x_0$, i.e. x_0 is a maximal element of X such that $x_0\succ z$.

The next lemma formalizes the use of the separation theorem in $E\times R$ referred to above. Recall that a function f is convex [concave, affine] if for each x, y in E and $0<\lambda<1$, $f(\lambda x+(1-\lambda)y)\leqq [\geqq,=]\ \lambda f(x)+(1-\lambda)f(y)$.

LEMMA 2. *Suppose that ψ_1 is a lower semicontinuous convex function on E with $-\infty<\psi_1\leqq\infty$ ($\psi_1\not\equiv\infty$), and that ψ_2 is a continuous concave function on E($\psi_2\not\equiv-\infty$). If $\psi_1\geqq\psi_2$, then there exists a continuous affine functional h on E such that $\psi_1\geqq h\geqq\psi_2$.*

Proof. In the space $E\times R$ (product topology) let $C_1=\{(x,r): r\geqq\psi_1(x)\}$ and let $C_2=\{(x,r): r\leqq\psi_2(x)\}$. Then C_1 and C_2 are closed convex sets and since ψ_2 is continuous, the interior of C_2 is the nonempty set $\{(x,r): r<\psi_2(x)\}$. This latter set misses C_1, so the separation theorem can be applied to yield a closed hyperplane H which separates C_1 and C_2. To see that H is the graph of a continuous affine function h (defined by $h(x)=r$ if and only if $(x,r)\in H$), we need only show that for at least one point x_0, the line $\{x_0\}\times R$ is not contained in H. Any point x_0 such that $\psi_1(x_0)<\infty$ will suffice, since $(x_0,\psi_1(x_0))\in C_1$ and will be on one side of H while $(x_0,\psi_2(x_0)-1)\in \operatorname{int} C_2$ and will be strictly on the other side. Finally, for any x, $(x,h(x))\in H$, so if $\psi_1(x)<h(x)$ then $(x,\psi_1(x))$ will be strictly on one side of H, and in C_1. For $r>h(x)>\psi_1(x)$, (x,r) is strictly on the other side of H and also in C_1, an impossibility which implies that $h\leqq\psi_1$. Similarly, $\psi_2\leqq h$ and the proof is complete.

The functions ψ_1 and ψ_2 which we will use in the proofs are certain combinations of linear functionals, the norm, and one other classical convex function, the *support functional* s_C for a weak* closed convex subset C of E^*. For x in E define $s_C(x)=\sup\{f(x):f\in C\}\equiv\sup C(x)$. This function is clearly convex and positive-homogeneous (i.e. $s_C(\lambda x)=\lambda s_C(x)$ if $\lambda\geqq 0$), and it follows from the separation theorem (applied in E^* in its weak* topology) that $f\in C$ if (and only if) $f\leqq s_C$. It is not difficult to see that s_C is lower semicontinuous. Finally, if C is *bounded*, with $\|f\|\leqq M$, say, for $f\in C$, then for any such f and any x, y in E, we have $f(x)=f(x-y)+f(y)\leqq\sup C(x-y)+s_C(y)\leqq M\|x-y\|+s_C(y)$. Thus, $|s_C(x)-s_C(y)|\leqq M\|x-y\|$ for all x, y, so that s_C is (uniformly) continuous on E.

THEOREM 1. *If C is a weak* closed convex subset of E^*, then the weak* support points of C are norm-dense in the norm-boundary of C.*

Proof. Given f in the boundary of C and $\varepsilon>0$, choose f_1 in $E^*\sim C$ such that $\|f-f_1\|<\varepsilon/2$ and use the separation theorem to choose z in E, $\|z\|=1$, such that $\sup C(z)<f_1(z)$. Then $f(z)\geqq f_1(z)-\varepsilon/2>\sup C(z)-\varepsilon/2=s_C(z)-\varepsilon/2$ and $f\leqq s_C$. Let $\phi=f-s_C$, so that ϕ is upper semicontinuous and $\phi\leqq 0$. By applying Lemma 1 (with $k=\varepsilon$) we can obtain x_0 in E such that $x_0\succ z$ and x_0 is maximal in the (nonempty) set where $\phi>-\infty$. The first assertion implies that $\varepsilon\|x_0-z\|\leqq\phi(x_0)-\phi(z)\leqq-\phi(z)=s_C(z)-f(z)\leqq\varepsilon/2$, so $\|x_0-z\|\leqq 1/2$, and hence (recall that $\|z\|=1$) $x_0\neq 0$. The maximality of x_0 implies that for any $y\neq x_0$ such that $\phi(y)>-\infty$ (equivalently, $s_C(y)<\infty$) we have $\varepsilon\|y-x_0\|>$ $>\phi(y)-\phi(x_0)=f(y-x_0)-[s_C(y)-s_C(x_0)]$. Let $\psi_1=s_C-s_C(x_0)$, $\psi_2(y)=$ $=f(y-x_0)-\varepsilon\|y-x_0\|$. By Lemma 2 there exist a continuous affine functional h on E such that $\psi_1\geqq h\geqq\psi_2$. Since

$$0=\psi_1(x_0)\geqq h(x_0)\geqq\psi_2(x_0)=0,$$

there exists a continuous *linear* functional g on E such that $h=g-g(x_0)$. Thus $s_C(y)-g(y)\geqq s_C(x_0)-g(x_0)$ for any y; since s_C and g are positive-homogeneous, this implies that $s_C-g\geqq 0$ and $s_C(x_0)-g(x_0)=0$, i.e. $g\in C$ and $g(x_0)=\sup C(x_0)$. The inequality $h\geqq\psi_2$ means, on the other hand, that for all y, $g(y-x_0)\geqq$ $\geqq f(y-x_0)-\varepsilon\|y-x_0\|$. Thus, for any x, $(f-g)(x)\leqq\varepsilon\|x\|$, so that $\|f-g\|\leqq\varepsilon$ and the proof is complete.

THEOREM 2. *Suppose that C and A are weak* closed convex subsets of E^*, that A is bounded, and that for some x in E, $\sup C(x)<\inf A(x)$. Then for any*

$\varepsilon > 0$ *there exists* x_0 *in* E *which supports* C *such that* $\|x - x_0\| < \varepsilon$ *and* $\sup C(x_0) < \inf A(x_0)$.

Proof. Choose f in C such that $f(x) > \sup C(x) - 1$. Let

$$B = -A(= \{-g: g \in A\});$$

we can assume without loss of generality that $0 \in A$ so that $s_B \geqq 0$. Since $f \in C$, $f - s_C \leqq 0$, and hence $0 \geqq \phi = f - s_C - s_B$. Furthermore, $\phi(x) > -1 - s_B(x)$. Since ϕ is upper semicontinuous, it follows from Lemma 1 (with $k > 0$ to be chosen later) that there exists x_0 in E such that $k\|x_0 - x\| \leqq \phi(x_0) - \phi(x) \leqq$ $\leqq -\phi(x) < 1 + s_B(x)$ and $k\|y - x_0\| > \phi(y) - \phi(x_0)$ for y in E, $y \neq x_0$. Let $\psi_1 = s_C - s_C(x_0)$ and $\psi_2(y) = f(y - x_0) - k\|y - x_0\| - [s_B(y) - s_B(x_0)]$; then ψ_1 and ψ_2 satisfy the hypotheses to Lemma 2 and $\psi_1(y) > \psi_2(y)$ if $y \neq x_0$. Thus, there exists a continuous affine functional h on E such that $\psi_1 \geqq h \geqq \psi_2$. Exactly as in the proof of Theorem 1, we have $h = g - g(x_0)$ for some continuous linear functional g on E, and $s_C - g \geqq 0 = s_C(x_0) - g(x_0)$, so that $g \in C$ and $g(x_0) = \sup C(x_0)$. Since $k\|x_0 - x\| \leqq 1 + s_B(x)$, for sufficiently large k we have $\|x_0 - x\| < \varepsilon$. Finally, since $\phi(x_0) - \phi(x) \geqq k\|x_0 - x\| \geqq 0$, we have $f(x_0) - s_C(x_0) - s_B(x_0) \geqq f(x) - s_C(x) - s_B(x) = f(x) + \beta$, where $\beta = -s_C(x) - s_B(x) = -\sup C(x) + \inf A(x)$ is positive by hypothesis. Thus, $\inf A(x_0) - \sup C(x_0) = -s_C(x_0) - s_B(x_0) \geqq f(x - x_0) + \beta \geqq \beta - \|f\|\|x - x_0\| \geqq \beta - \|f\| k^{-1}[1 + s_B(x)]$.

For sufficiently large k this is positive and the proof is complete.

We conclude with some examples which show that Theorem 1 may fail if certain of the hypotheses are weakened. For instance, it is not enough to assume merely that C is norm closed. Indeed, suppose that E is not reflexive and choose an element F in E^{**}, $\|F\| = 1$, which is not in the canonical image of E. Let $C = \{f: f \in E^*, \|f\| \leqq 1 \text{ and } |F(f)| \leqq 1/2\}$; clearly, C is norm closed, convex, bounded and has nonempty norm-interior. It is easily verified that the set $A = \{f: F(f) = 1/2 \ \|f\| < 1\}$ is open in the norm boundary of C, but contains no weak* support points of C. (Any functional which supports C at a point of A is necessarily a constant multiple of F, hence not weak* continuous.)

In Theorem 3 of [1] it is shown that in any *incomplete* normed linear space E there exists a symmetric bounded closed convex set A with nonempty interior such that the support functionals of A are not dense in E^* (hence those which lie in the boundary B of the polar A° of A are not dense in B). The polar $C = A^\circ$ of A is therefore a weak* compact convex subset of E^*, but its weak* support points are not dense in its norm-boundary, which shows that completeness of E is essential to Theorem 1. (It is interesting to note that we do *not* require completeness of E^* in the weak* topology; this occurs only when E is finite dimensional.)

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